Weird Inequalities

Black Lecture, June 29

1. (IMO 2003/05) Let $x_1 \le x_2 \le \cdots \le x_n$ be real numbers. Prove that

$$\Big(\sum_{i,j=1}^{n} |x_i - x_j|\Big)^2 \le \frac{2}{3}(n^2-1)\sum_{i,j=1}^{n}(x_i - x_j)^2$$

and determine when equality occurs.

2. (IMO 1999/2) Let $n \ge 2$ be a fixed integer. Find the smallest constant C such that for all non-negative reals $x_1, x_2, \ldots, x_n$,

$$\sum_{1\le i<j\le n} x_i x_j (x_i^2 + x_j^2) \le C\Big(\sum_{i=1}^{n} x_i\Big)^4.$$

3. (Russia 2004) Let $n > 3$ be an integer and let $x_1, x_2, \ldots, x_n$ be positive reals with product 1. Prove that

$$\frac{1}{1+x_1+x_1x_2} + \frac{1}{1+x_2+x_2x_3} + \cdots + \frac{1}{1+x_n+x_nx_1} > 1.$$

4. (Romania 2004) Let $n \ge 2$ be an integer and let $a_1, a_2, \ldots, a_n$ be real numbers. Prove that for any non-empty subset $S \subset \{1, 2, 3, \ldots, n\}$, we have

$$\Big(\sum_{i\in S} a_i\Big)^2 \le \sum_{1\le i\le j\le n}(a_i + \cdots + a_j)^2.$$

5. (Romania 1996) Let $x_1, \ldots, x_{n+1}$ be positive real numbers such that $x_1 + \cdots + x_n = x_{n+1}$. Prove that

$$\sum_{i=1}^{n}\sqrt{x_i(x_{n+1}-x_i)} \le \sqrt{\sum_{i=1}^{n} x_{n+1}(x_{n+1}-x_i)}.$$

6. (IMO 2001 short list) Let $x_1, x_2, \ldots, x_n$ be arbitrary real numbers. Prove the inequality

$$\frac{x_1}{1+x_1^2} + \frac{x_2}{1+x_1^2+x_2^2} + \cdots + \frac{x_n}{1+x_1^2+\cdots+x_n^2} < \sqrt{n}.$$

7. Let $x_1, x_2, \ldots, x_n$ be positive real numbers with $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n}\frac{1}{x_i}$. Prove that

$$\sum_{i=1}^{n}\frac{1}{n-1+x_i} \le 1.$$

8. (IMO 2004 short list) Let n be a positive integer and let $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ be two sequences of positive real numbers. Suppose $(z_2, \ldots, z_{2n})$ is a sequence of positive real numbers such that $z_{i+j}^2 \ge x_i y_j$ for all $1 \le i, j \le n$. Let $M = \max\{z_2, \ldots, z_{2n}\}$. Prove that

$$\Big(\frac{M + z_2 + \cdots + z_{2n}}{2n}\Big)^2 \ge \Big(\frac{x_1+\cdots+x_n}{n}\Big)\Big(\frac{y_1+\cdots+y_n}{n}\Big).$$